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# Reduction of the representations of the generalised Poincaré algebra by the Galilei algebra

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**Abstract.** The realisations of all classes of unitary irreducible representations of the generalised Poincaré group P(1,4) have been found in a basis in which the Casimir operators of its important subgroup, i.e. the Galilei group, are of diagonal form. The exact form of the unitary operator which connects the canonical basis of the P(1,4) group and the Galilei basis has been established.

## 1. Introduction

Some years ago it was proposed to use the generalised Poincaré group P(1,4), the group of displacements and rotations in five-dimensional Minkovsky space, for the description of particles with variable masses and spins (Fushchich and Krivsky 1968a, b, Fushchich 1970). This and other generalised groups P(1,*n*), P(2,3) etc were considered and used successively by Castell (1967), Aghassi *et al* (1970), Barrabes and Henry (1976), Elizalde and Gomish (1978) and many others.

The main property of the P(1,4) group is that it contains the Poincaré group P(1,3) as well as the Galilei group G(3) as its subgroups†. So the P(1,4) group unified the groups of motion of relativistic and non-relativistic quantum mechanics.

For the elucidation of the physical grounds of the generalised quantum mechanics based on the P(1,4) group (Fushchich and Krivsky 1968a, b, 1969) the important problem is the reduction of the irreducible representations IR of the P(1,4) group, or the Lie algebra of the P(1,4) group, by the IR of its subgroups, or its subalgebras‡. The problem of the reduction of IR of the P(1,4) algebra corresponding to the time-like five-momenta by its subalgebra P(1,3) has been solved (Fushchich *et al* 1976, Nikitin *et al* 1976), i.e. the type of representations of the P(1,3) algebra contained in the IR of the P(1,4) algebra has been investigated and the unitary operator was found which connects the canonical basis of the P(1,4) group representation with the P(1,3) basis, in which the Casimir operators of the Poincaré group have the diagonal form (the spectrum of these operators is nondegenerate).

In this paper we find the realisation of the IR of the P(1,4) algebra in the 'Galilei basis' namely, in the basis in which the invariant operators of the Galilei subalgebra are diagonal ones. We also obtain the explicit form of the unitary operator, which carries

† The paper of Fedorchuck (1978) is devoted to the classification and the description of all subgroups of the P(1,4) group.

‡ We will indicate the groups and the corresponding Lie algebras by the same indices.

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out the reduction $P(1,3) \rightarrow G(2)$ which plays an important role in the null-plane approach (see e.g. Leutwyler and Stern 1968).

## 2. Statement of the problem

The Lie algebra of the P(1,4) group is specified by the fifteen generators $P_\mu$, $J_{\mu\nu}$ ($\mu$, $\nu = 0, 1, 2, 3, 4$), which satisfy the commutation relations

$$\begin{aligned} &[P_\mu, P_\nu] = 0 \qquad [P_\mu, J_{\nu\sigma}] = \mathrm{i}(g_{\mu\nu}P_\sigma - g_{\mu\sigma}P_\nu) \\ &[J_{\mu\nu}, J_{\rho\sigma}] = \mathrm{i}(g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho}). \end{aligned} \tag{2.1}$$

The algebra (2.1) has three main invariant (Casimir) operators (Fushchich and Krivsky 1968a, b)

$$P^2 = P_\mu P^\mu = P_0^2 - \boldsymbol{P}^2 - P_4^2 \qquad V_1 = \tfrac{1}{2}\omega_{\mu\nu}\omega_{\mu\nu} \qquad V_2 = -\tfrac{1}{4}J_{\mu\nu}\omega_{\mu\nu} \tag{2.2}$$

where

$$\omega_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma\lambda}J^{\rho\sigma}P^\lambda.$$

As in the case of the Poincaré group, one can specify four different classes of the representations of the algebra (2.1), corresponding to $P^2 > 0$, $P^2 = 0$, $P^2 < 0$ and $P_\mu \equiv 0$ (in the last case one arrives at the representations of the homogeneous group SO(1,4), which are not considered here).

Algebra (2.1) contains the Lie algebras of the Poincaré and of the Galilei groups as subalgebras. In order to select the subalgebra P(1,3) it is enough to consider the relations (2.1) for $\mu$, $\nu \neq 4$. The subalgebra G(3) may be obtained by the transition to the new basis

$$\begin{aligned} &\hat{P}_0 = \tfrac{1}{2}(P_0 - P_4) \qquad M = P_0 + P_4 \qquad \hat{P}_a = P_a \qquad K = J_{0a} \\ &J_a = \tfrac{1}{2}\epsilon_{abc}J_{bc} \qquad G_a^+ = J_{0a} + J_{4a} \qquad G_a^- = \tfrac{1}{2}(J_{0a} - J_{4a}). \end{aligned} \tag{2.3}$$

The operators (2.3) satisfy the commutation relations

$$\begin{aligned} &[\hat{P}_0, P_a] = [\hat{P}_0, M] = [\hat{P}_a, M] = [\hat{P}_a, \hat{P}_b] = 0 \\ &[\hat{P}_0, J_a] = [M, J_a] = [G_a^+, G_b^+] = [M, G_a^+] = 0 \\ &[\hat{P}_a, J_b] = \mathrm{i}\epsilon_{abc}\hat{P}_c \qquad [G_a^+, \hat{P}_b] = \mathrm{i}\delta_{ab}M \\ &[J_a, J_b] = \mathrm{i}\epsilon_{abc}J_c \qquad [\hat{P}_0, G_b^+] = \mathrm{i}\hat{P}_b \end{aligned} \tag{2.4}$$

$$\begin{aligned} &[\hat{P}_0, G_a^-] = [G_a^-, G_b^-] = 0 \qquad [G_a^-, M] = -\mathrm{i}\hat{P}_a \\ &[G_a^-, J_b] = \mathrm{i}\epsilon_{abc}G_c^- \qquad [G_a^-, \hat{P}_b] = -\mathrm{i}\delta_{ab}\hat{P}_0 \\ &[G_a^+, G_a^-] = \mathrm{i}(\epsilon_{abc}J_c + \delta_{ab}K) \\ &[\hat{P}_a, K] = [J_a, K] = 0 \qquad [\hat{P}_0, K] = -\mathrm{i}P_0 \\ &[M, K] = \mathrm{i}M \qquad [G_a^\pm, K] = \pm\mathrm{i}G_a^\pm. \end{aligned} \tag{2.5}$$

The commutation relations (2.4) specify the Lie algebra of the extended Galilei group (Bargman 1954). The invariant operators of this algebra are given by the formulae

$$C_1 = 2M\hat{P}_0 - \hat{\boldsymbol{P}}^2 \qquad C_2 = (M\boldsymbol{J} - \hat{\boldsymbol{P}} \times \boldsymbol{G}^+)^2 \qquad C_3 = M. \tag{2.6}$$

Our aim is to find the realisations of the generators (2.3) for any class of IR of the P(1,4) algebra, in a basis where the Casimir operators (2.6) have a diagonal form. This enables us to answer the question what IR of the G(3) algebra are contained in the given representation of the P(1,4) algebra and to establish the connection between the vectors in the Poincaré and in the Galilei bases.

The realisations of all IR of the P(1,4) algebra have already been found (Fushchich and Krivsky 1968a, b, 1969, Fushchich 1970). So the problem of the description of the IR of the P(1,4) algebra in the Galilei basis reduces to transforming the known realisation to a form in which the operators (2.6) are diagonal.

## 3. The representations with $P^2 \geqslant 0$

Let us consider the IR of the P(1,4) algebra, which corresponds to the positive values of the invariant operator $P^2 = \varkappa^2 > 0$. The generators $P_\mu$, $J_{\mu\nu}$ in the canonical basis $|p_k, j_3, \tau_3; \epsilon, j, \tau, \varkappa\rangle$ have the form (Fushchich and Krivsky 1968a, b)

$$\begin{aligned} &P_0 = \epsilon E \equiv \epsilon(p^2 + p_4^2 + \varkappa^2)^{1/2} \qquad && P_k = p_k \\ &J_{kl} = \mathrm{i}\left(p_l \frac{\partial}{\partial p_k} - p_k \frac{\partial}{\partial p_l}\right) + S_{kl}, && k, l = 1, 2, 3, 4 \\ &J_{0k} = -\mathrm{i}\epsilon E \frac{\partial}{\partial p_k} - \epsilon \frac{S_{kl} p_l}{E + \varkappa} && \epsilon = \pm 1 \end{aligned} \tag{3.1}$$

where $S_{kl}(k, l = 1, 2, 3, 4)$ are the generators of the IR $D(j, \tau)$ of the SO(4) group.

The basis of the realisation (3.1) is formed by the vectors $|p_k, j_3, \tau_3; \epsilon, j, \tau, \varkappa\rangle$, which are the eigenfunctions of the complete set of the commuting operators

$$T = P_k \qquad J_3 = \tfrac{1}{2}(\omega_{12} + \omega_{43}) \qquad T_3 = \tfrac{1}{2}(\omega_{12} - \omega_{43}) \qquad \hat{\epsilon} = P_0/|P_0|$$

$$J^2 = \frac{1}{4\varkappa^2}(V_1 + 2\epsilon\varkappa V_2) \qquad T^2 = \frac{1}{4\varkappa^2}(V_1 - 2\epsilon\varkappa V_2) \qquad P^2,$$

with the eigenvalues $p_k$, $j_3$, $\tau_3$, $\epsilon$, $j(j+1)$, $\tau(\tau+1)$ and $\varkappa^2$ correspondingly, where $j$ and $\tau$ are the integers or half-integers labelling the IR of the SO(4) group,

$$\begin{aligned} &j_3 = -j, -j+1, \ldots, j \qquad && \tau_3 = -\tau, -\tau+1, \ldots, \tau \\ &\epsilon = \pm 1 && -\infty < p_k < \infty. \end{aligned}$$

The basis vectors may be normalised according to

$$\langle p_k, j_3, \tau_3; \epsilon, j, \tau, \varkappa | p_k', j_3', \tau_3'; \epsilon, j, \tau, \varkappa\rangle = 2E\delta(p_k - p_k')\delta_{j_3 j_3'}\delta_{\tau_3\tau_3'},$$

and the generators (3.1) are Hermitian with respect to the scalar product

$$(\Psi_1, \Psi_2) = \int (\mathrm{d}^4 p/E)\Psi_1^\dagger(p_k, j_3, \tau_3)\Psi_2(p_k, j_3, \tau_3). \tag{3.2}$$

The basis of the IR of the P(1,4) algebra, in which the invariant operators (2.6) of the G(3) algebra and the operators $P_a (a = 1, 2, 3)$ and $S_3 = J_3 - (1/m)(P_2 G_1^+ - P_1 G_2^+)$ have the diagonal form, will be called 'Galilei basis' (or 'G(3) basis') and denoted by $|p_a, m, s, s_3; \epsilon, j, \tau, \varkappa\rangle$.

We will normalise the basis vectors as

$$\langle p_a, m, s, s_3; \epsilon, j, \tau, \varkappa | p'_a, m', s', s'_3; \epsilon, j, \tau, \varkappa\rangle = 2m\delta(m-m')\delta(p_a-p'_a)\delta_{ss'}\delta_{s_3s'_3}.$$

This will lead us to the scalar product

$$(\phi_1, \phi_2) = \sum_{|j-\tau|\leqslant s\leqslant j+\tau} \int_\varkappa^\infty \frac{\mathrm{d}m}{m} \int \mathrm{d}^3p\, \phi_1^\dagger(s, s_3, m, \boldsymbol{p})\phi_2(s, s_3, m, \boldsymbol{p}). \tag{3.3}$$

Our task is to establish the explicit form of the generators of the P(1,4) group in the Galilei basis and to find the transition operator, which connects the canonical and Galilei bases. First we substitute (3.1) into (2.3) and (2.6) and obtain the Galilei generators $\hat{P}_\mu$, $J_a$, $G_a^+$, the invariant operators $C_a$ and the remaining generators $G_a^-$, $K$ in the canonical basis in a form

$$\hat{P}_0 = \tfrac{1}{2}(\epsilon E - p_4) \qquad M = \epsilon E + p_4 \qquad J_a = -\mathrm{i}(\boldsymbol{p}\times(\partial/\partial\boldsymbol{p}))_a + S_a$$
$$G_a^+ = x_4 p_a - Mx_a - \frac{\epsilon S_{ab}p_b - S_{4a}(E+\varkappa+\epsilon p_4)}{E+\varkappa} \tag{3.4}$$

$$C_1 = \varkappa^2 \qquad C_2 = \{S^2[M(E+\varkappa)-\epsilon p^2]^2 + [p^2\mathcal{N}^2 - (\boldsymbol{p}\cdot\mathcal{N})^2](E+\varkappa+\epsilon p_4)^2$$
$$+(\boldsymbol{p}\cdot\boldsymbol{S})^2[2\epsilon M(E+\varkappa)-\boldsymbol{p}^2]\}(E+\varkappa)^{-2} \qquad C_3 = M \tag{3.5}$$

$$G_a^- = \tfrac{1}{2}\left[-x_4p_a - \hat{P}_0x_a - \frac{\epsilon S_{ab}p_b - S_{4a}(E+\varkappa-\epsilon p_4)}{E+\varkappa}\right]$$
$$K = -\hat{P}_0x_4 - \epsilon\frac{S_{4a}p_a}{E+\varkappa}, \tag{3.6}$$

where

$$S_a = \tfrac{1}{2}\epsilon_{abc}S_{bc} \qquad \mathcal{N}_a = S_{4a} \qquad x_k = \mathrm{i}(\partial/\partial p_k). \tag{3.7}$$

The Casimir operator $C_2$ (3.5) is in general the matrix which has elements depending on $p_k$. Our second step is to diagonalise this matrix with the help of some unitary transformation. We will look for the diagonalising operator in a form

$$U_1 = \exp(\mathrm{i}S_{4a}p_a\theta/p) \tag{3.8}$$

where $p = (p_1^2+p_2^2+p_3^2)^{1/2}$ and $\theta$ is an unknown function of $p$, $p_4$.

With the help of the operator (3.8) one may derive from (3.4) and (3.6) a new realisation:

$$\hat{P}'_0 = U_1\hat{P}_0U_1^\dagger = \hat{P}_0 \qquad \hat{P}'_a = U_1\hat{P}_aU_1^\dagger = \hat{P}_a$$
$$J'_a = U_1J_aU_1^\dagger = J_a \qquad M' = U_1MU_1^\dagger = M \tag{3.9}$$

$$(G_a^+)' = U_1G_a^+U_1^\dagger = x'_4p_a - x'_aM - \frac{\epsilon S'_{ab}p_b - S'_{4a}(E+\varkappa+\epsilon p_4)}{E+\varkappa} \tag{3.10}$$

$$(G_a^-)' = U_1G_a^-U_1^\dagger = \tfrac{1}{2}\left(-x'_4p_a - \hat{P}_0x'_a - \frac{\epsilon S'_{ab}p_b - S'_{4a}(E+\varkappa-\epsilon p_4)}{E+\varkappa}\right)$$
$$K' = U_1KU_1^\dagger = -\hat{P}_0x'_4 - \epsilon S'_{4a}p_a/(E+\varkappa) \tag{3.11}$$

where

$$x'_k = U_1x_kU_1^\dagger \qquad S'_{kl} = U_1S_{kl}U_1^\dagger.$$

Using the Hausdorf–Campbell formula

$$\exp(A)B\exp(-A)=\sum_{n=0}^{\infty}\frac{1}{n!}\{A,B\}^n \qquad \{A,B\}^n=[\{A,B\}^{n-1}] \qquad \{A,B\}^0=B$$

it is not difficult to calculate

$$\begin{aligned}
x'_a &= x_a+(p_aS_{4b}p_b/p^2)[\partial\theta/\partial p-(\sin\theta)/p]+(S_{ab}p_b/p^2)(1-\cos\theta)+(1/p)S_{4a}\sin\theta\\
S'_{4a} &= S_{4a}\cos\theta+(p_aS_{4b}p_b/p^2)(1-\cos\theta)+S_{ab}p_b(\sin\theta)/p\\
S'_{ab}p_b &= S_{ab}p_b\cos\theta+[(p_aS_{4b}p_b/p)-pS_{4a}]\sin\theta \qquad x'_4=x_4+(S_{4b}p_b/p)(\partial\theta/\partial p_4).
\end{aligned}\tag{3.12}$$

Substituting (3.12) into (3.10), one obtains

$$\begin{aligned}
(G_a^+)' = x_4p_a-Mx_a &+\frac{p_aS_{4b}p_b}{p}\left[\frac{\partial\theta}{\partial p_4}-\frac{M}{p}\left(\frac{\partial\theta}{\partial p}-\frac{1}{p}\sin\theta\right)-\frac{\epsilon}{E+\varkappa}\sin\theta\right.\\
&\left.+\frac{E+\varkappa+\epsilon p_4}{(E+\varkappa)p}(1-\cos\theta)\right]+\frac{S_{ab}p_b}{p}\left[\left(\frac{M}{p}-\frac{\epsilon p}{E+\varkappa}\right)\cos\theta-\frac{M}{p}\right.\\
&\left.+\frac{E+\varkappa+\epsilon p_4}{(E+\varkappa)}\sin\theta\right]+S_{4a}\left[\left(\frac{\epsilon p}{E+\varkappa}-\frac{M}{p}\right)\sin\theta+\frac{E+\varkappa+\epsilon p_4}{(E+\varkappa)}\cos\theta\right].
\end{aligned}\tag{3.13}$$

The expression (3.13) for $G_a^+$ is much simplified, if one puts

$$\theta=2\tan^{-1}[p/(E+\epsilon p_4+\varkappa)].\tag{3.14}$$

For such a value of the parameter $\theta$, we have:

$$\sin\theta=\frac{p(E+\varkappa+\epsilon p_4)}{(E+\varkappa)(E+\epsilon p_4)} \qquad 1-\cos\theta=[p^2/(E+p)(E+\epsilon p_4)]$$

$$\epsilon\frac{\partial\theta}{\partial p_4}-\frac{E+\epsilon p_4}{p}\frac{\partial\theta}{\partial p}=-\sin\theta\frac{E+\epsilon p_4}{p^2}$$

and

$$(G_a^+)'=x_4p_a-Mx_a.\tag{3.15}$$

Substituting (3.9) and (3.15) into (2.6), we have

$$C'_2=M^2\boldsymbol{S}^2\tag{3.16}$$

where the matrix $\boldsymbol{S}^2=S_1^2+S_2^2+S_3^2$ always may be chosen in the diagonal form,

$$\boldsymbol{S}^2\phi_s=s(s+1)\phi_s \qquad |j-\tau|\leqslant s\leqslant j+\tau.$$

The operators (3.9)–(3.11) are defined in a Hilbert space of square integrable functions $\phi(p_1,p_2,p_3,p_4)$. In order to diagonalise the operator $M$ (3.4) and (3.5) we introduce in place of $\{p_1,p_2,p_3,p_4\}$ the new variables $\{p_1,p_2,p_3,m\}$ where $m=E+\epsilon p_4$. Then

$$\frac{\partial}{\partial p_4}\to\left(\epsilon+\frac{p_4}{E}\right)\frac{\partial}{\partial m} \qquad \frac{\partial}{\partial p_a}\to\frac{\partial}{\partial p_a}+\frac{p_a}{E}\frac{\partial}{\partial m}$$

and the operators (3.9)–(3.11) and (3.15) take the form

$$\hat{P}'_0 = m_0 + \epsilon\frac{p^2}{2m} \qquad \hat{P}'_a = p_a \qquad M' = \epsilon m$$

$$J'_a = -\mathrm{i}(\boldsymbol{p}\times(\partial/\partial\boldsymbol{p}))_a + S_a \qquad (G^+_a)' = -\mathrm{i}\epsilon m(\partial/\partial p_a) \tag{3.17a}$$

$$C'_1 = \varkappa^2 \qquad C'_2 = m^2\boldsymbol{S}^2 \qquad C'_3 = \epsilon m \tag{3.17b}$$

$$K' = -\mathrm{i}m(\partial/\partial m) \qquad (G^-_a)' = \mathrm{i}[\epsilon p_a(\partial/\partial m) - \hat{P}'_0(\partial/\partial p_a)] - \epsilon(S_{ab}p_b + S_{4a}\varkappa)/m \tag{3.17c}$$

where

$$\varkappa \leqslant m < \infty \qquad m_0 = \epsilon(\varkappa^2/2m).$$

The generators (3.17) are Hermitian with respect to the scalar product (3.3).

So we reach the following result:

*Theorem.* The Hilbert space of the IR $D^\epsilon(\varkappa, j, \tau)$ of the P(1,4) algebra, corresponding to $P^2 = \varkappa^2 > 0$, is expanded into the direct integral of the subspaces, which correspond to the IR of the G(3) algebra with the following values of the invariant operators: $C_1 = \varkappa^2$, $C_2 = m^2 s(s+1)$, $C_3 = \epsilon m$, $|\varkappa| \leqslant m < \infty$, $|j-\tau| \leqslant S \leqslant j+\tau$. The explicit form of the P(1,4) group generators in the Galilei basis and that of the transition operator, which connects the canonical and the G(3) bases, are given by the formulae (3.8), (3.14) and (3.17).

To conclude this section we consider the IR of the P(1,4) algebra, corresponding to $P^2 = 0$. The realisations of such an IR have been obtained in the form (Fushchich and Krivsky 1968a, b):

$$P_0 = \epsilon E_0 \equiv \epsilon(p^2 + p_4^2)^{1/2} \qquad P_a = p_a \qquad P_4 = p_4$$

$$J_{0a} = -\mathrm{i}\epsilon E_0\frac{\partial}{\partial p_a} - \epsilon\frac{S_{ab}p_b}{E_0 + p_4} \qquad J_{04} = -\mathrm{i}\epsilon E_0\frac{\partial}{\partial p_4}$$

$$J_{4a} = \mathrm{i}\left(p_a\frac{\partial}{\partial p_4} - p_4\frac{\partial}{\partial p_a}\right) + \epsilon\frac{S_{ab}p_b}{E_0 + p_4}$$

where $S_{ab}$ are the generators of the IR $D(s)$ of the SO(3) group. Substituting (3.18) into (2.3), one obtains

$$\hat{P}_0 = \tfrac{1}{2}(\epsilon E_0 - p_4) \qquad M = \epsilon E_0 + p_4$$

$$J_a = -\mathrm{i}\left(\boldsymbol{p}\times\frac{\partial}{\partial\boldsymbol{p}}\right)_a + S_a$$

$$G^+_a = \mathrm{i}\left(p_a\frac{\partial}{\partial p_4} - p_4\frac{\partial}{\partial p_a}\right) + \mathrm{i}\epsilon E_0\frac{\partial}{\partial p_a} \tag{3.18}$$

$$K = -\mathrm{i}\epsilon E_0\frac{\partial}{\partial p_4}$$

$$G^-_a = \tfrac{1}{2}\left(-\mathrm{i}p_a\frac{\partial}{\partial p_4} - \mathrm{i}\hat{P}_0\frac{\partial}{\partial p_a}\right) - \epsilon\frac{S_{ab}p_b}{E_0 + \epsilon p_4}.$$

It is not difficult to see that replacement of the variables $\{\boldsymbol{p}, p_4\} \to \{\boldsymbol{p}, m\}$, where $m = E_0 + \epsilon p_4$, reduces the generators (3.18) to the form (3.17), where, however, $\varkappa = 0$, $0 \leqslant m < \infty$ and $s$ has the fixed value, which characterises the IR of the SO(3) group. So we have established the explicit form of the generators of the P(1,4) group, corresponding to $P^2 = 0$, in the Galilei basis.

## 4. The representations with $P^2 < 0$

We now use the IR of the P(1,4) group, which corresponds to $P^2 = -\eta^2 < 0$. The generators of such representations have been obtained in the form (Fushchich and Krivsky 1968a, b, 1969)

$$\begin{aligned} &P_0 = p_0 \qquad P_a = p_a \qquad P_4 = \epsilon(p_0^2 + \eta^2 - p_a^2)^{1/2} \\ &J_{\alpha\beta} = \mathrm{i}\left(p_\beta \frac{\partial}{\partial p_\alpha} - p_\alpha \frac{\partial}{\partial p_\beta}\right) + S_{\alpha\beta} \qquad \epsilon = \pm 1 \\ &J_{4\alpha} = -\mathrm{i}P_4 \frac{\partial}{\partial p_\alpha} - \epsilon \frac{S_{\alpha\beta} p^\beta}{|P_4| + \eta} \qquad \alpha, \beta = 0, 1, 2, 3, \end{aligned} \tag{4.1}$$

where $S_{\alpha\beta}$ are the matrices which realise IR of the Lie algebra of the SO(1,4) group.

Reducing the representation (4.1) by the representations of the Lie algebra of the Galilei group, the mass operator $M = P_0 + P_4$ may take the zero value. Let us impose the G(3)-invariant condition of turning into zero in the hyperspace, corresponding to zero eigenvalues of the operator $M$, on the functions from the space of the IR (4.1) (this hyperspace is the five-dimensional half-cylinder $p^2 = \eta^2$, $\epsilon p_0 < 0$).

Using the transformation operator on the generators (4.1)

$$U_2 = \exp(\mathrm{i}S_{0a} p_a \theta / p) \qquad \theta = 2 \tanh^{-1}[p/(\eta + |P_4| + \epsilon p_0)] \tag{4.2}$$

and using the relations

$$U_2 x_0 U_2^{-1} = x_0 + S_{0a} p_a \frac{1}{p} \frac{\partial \theta}{\partial p_0} \qquad x_\mu = \mathrm{i} \frac{\partial}{\partial p_\mu}$$

$$U_2 x_a U_2^{-1} = x_a + \frac{p_a}{p} \frac{S_{0b} p_b}{p} \left(\frac{\partial \theta}{\partial p} - \frac{1}{p} \sinh \theta\right) + \frac{1}{p} S_{0a} \sinh \theta + \frac{S_{ab} p_b}{p^2} (1 - \cosh \theta)$$

$$U_2 S_{0a} U_2^{-1} = S_{0a} \cosh \theta - (1/p) S_{ab} p_b \sinh \theta + (p_a/p)(S_{0b} p_b / p)(1 - \cosh \theta)$$

$$U_2 S_{ab} p_b U_2^{-1} = S_{ab} p_b \cosh \theta + [(p_a S_{0b} p_b / p) - p S_{0a}] \sinh \theta$$

$$\sinh \theta = \frac{p(\epsilon p_0 + |P_4| + \eta)}{(\epsilon p_0 + |P_4|)(|P_4| + \eta)} \qquad \frac{\partial \theta}{\partial p_0} = \frac{p}{|P_4|(|P_4| + \eta)}$$

$$1 - \cosh \theta = \frac{-p^2}{(|P_4| + \eta)(\epsilon p_0 + |P_4|)} \qquad \frac{\partial \theta}{\partial p} = \frac{|P_4|(\epsilon p_0 + \eta) + p_0^2 + \eta^2}{|P_4|(|P_4| + \eta)(|P_4| + \epsilon p_0)},$$

one comes to the realisation

$$\begin{aligned} &P_0'' = p_0 \qquad P_a'' = p_a \qquad P_4'' = \epsilon(p_0^2 + \eta^2 - p^2)^{1/2} \\ &J_{ab}'' = \mathrm{i}\left(p_b \frac{\partial}{\partial p_a} - p_a \frac{\partial}{\partial p_b}\right) + S_{ab} \\ &J_{0a}'' = \mathrm{i}\left(p_a \frac{\partial}{\partial p_0} - p_0 \frac{\partial}{\partial p_a}\right) - \frac{S_{ab} p_b + S_{a0} \eta}{|P_4''| + \epsilon p_0} \\ &J_{4a}'' = -\mathrm{i}P_4'' \frac{\partial}{\partial p_a} + \frac{S_{ab} p_b + S_{a0} \eta}{|P_4''| + \epsilon p_0} \qquad J_{04}'' = \mathrm{i}P_4'' \frac{\partial}{\partial p_0}. \end{aligned} \tag{4.3}$$

Substituting (4.3) into (2.3) and going from $\{p_a, p_0\}$ to the new variables $\{p_a, m\}$, where $m = p_0 + (p_0^2 + \eta^2 - p_a^2)^{1/2}$, one obtains the Galilei group generators in the form (3.17*a*), and the remaining generators $G_a^-$, $K$ in the form (3.17*c*), where, however, $m_0 = -\eta^2/2m$, $-\eta^2 < m < 0$, $0 < m < \infty$, and $S_{ab}$ are the generators of the group $SO(3) \subset SO(1,3)$.

## 5. Covariant representation of the P(1,4) group

Consider an arbitrary covariant representation of the Lie algebra of the P(1,4) group. Such a representation is realised by the operators

$$P_\mu = p_\mu \qquad J_{\mu\nu} = \mathrm{i}\left(p_\nu \frac{\partial}{\partial p_\mu} - p_\mu \frac{\partial}{\partial p_\nu}\right) + S_{\mu\nu} \tag{5.1}$$

where $S_{\mu\nu}$ are the generators of a representation of the SO(1,4) group. Let us confine ourselves to the case where $P_\mu P^\mu \Psi > 0$.

Substituting (5.1) into (2.3), we obtain

$$\begin{aligned}
&\hat{P}_0 = \tfrac{1}{2}(p_0 - p_4) \qquad && \hat{P}_a = p_a \\
&J_a = -\mathrm{i}\left(\boldsymbol{p} \times \frac{\partial}{\partial \boldsymbol{p}}\right)_a + S_a && M = p_0 + p_4 \\
&G_a^+ = \tilde{x}_0 p_a - x_a M + \lambda_a^+ && \\
&G_a^- = \tilde{x}_4 p_a - \tilde{x}_a \hat{P}_0 + \tfrac{1}{2}\lambda_a^- && \\
&K = \tilde{x}_4 M - \tilde{x}_0 \hat{P}_0 + S_{04} &&
\end{aligned} \tag{5.2}$$

where

$$\lambda^\pm = S_{0a} \pm S_{4a} \qquad \tilde{x}_0 = 2\mathrm{i}\left(\frac{\partial}{\partial p_0} - \frac{\partial}{\partial p_4}\right) \qquad \tilde{x}_4 = \mathrm{i}\left(\frac{\partial}{\partial p_0} + \frac{\partial}{\partial p_4}\right).$$

For the transition of the realisation (5.2) into the Galilei basis we use the operator

$$U_3 = \exp[\mathrm{i}\boldsymbol{\lambda}^+ \boldsymbol{p}/M]. \tag{5.3}$$

With the help of the transformation

$$\begin{aligned}
&\hat{P}_\mu \to \hat{P}_\mu''' = U_3 \hat{P}_\mu U_3^{-1} \qquad && J_a \to J_a''' = U_3 J_a U_3^{-1} \\
&G_a^\pm \to (G_a^\pm)''' = U_3 G_a^\pm U_3^{-1} && K \to K''' = U_3 K U_3^{-1},
\end{aligned}$$

one comes to the realisation in which the invariant operators (2.6) of the G(3) subalgebra are of diagonal form:

$$\hat{P}_0''' = \tfrac{1}{2}(p_0 - p_4) \qquad \hat{P}_a''' = p_a \qquad M''' = M = p_0 + p_4$$

$$J_a''' = -\mathrm{i}(\boldsymbol{p} \times \partial/\partial \boldsymbol{p})_a + S_a \qquad G_a^+ = \tilde{x}_0 p_a - x_a M$$

$$G_a^- = \tilde{x}_4 p_a - x_a \hat{P}_0''' - \frac{S_{ab} p_b + S_{40} p_a}{M} + \tfrac{1}{2}\lambda_a^- - \lambda^+ \frac{p_\mu p^\mu}{M^2} \qquad K''' = \tilde{x}_4 M - \tilde{x}_0 \hat{P}_0''' + S_{04},$$

where $S_a = \frac{1}{2}\epsilon_{abc} S_{bc}$. The operators $C_a$ (2.6) take the form

$$C_1''' = p_\mu p^\mu \qquad C_2''' = M^2 \boldsymbol{S}^2 \qquad C_3''' = M$$

i.e. the eigenvalues of the operator $C_1$ coincide with the values of $P^2$, the eigenvalues of the operator $C_2$ are characterised by the spectrum of the Casimir operator of the group $SO(3) \subset SO(1,4)$, and the eigenvalues of the operator $C_3$ lie in the interval $(C_1''')^{1/2} \leqslant C_3''' < \infty$.

The results of this section may be used for the diagonalisation of the wave equations, which are invariant under the P(1,4) group. As an example we will consider the five-dimensional generalisation of the Dirac equation

$$(\gamma_\mu p^\mu + \varkappa)\Psi = 0 \qquad \mu = 0, 1, 2, 3, 4. \tag{5.4}$$

On the set of the solutions of the equation (5.4) the generators of the P(1,4) group have the form (5.1) where $S_{\mu\nu} = \frac{1}{4}\mathrm{i}[\gamma_\mu, \gamma_\nu]$. Using the operator (5.3) on equation (5.4), one obtains an equation, which is equivalent to (5.4) but is manifestly invariant under the Galilei group

$$\hat{P}_0''' \Phi_+ = (\varkappa/2m + p^2/2m)\Phi_+ \qquad \Phi_- = 0 \tag{5.5}$$

where

$$\Phi_\pm = \tfrac{1}{2}(1 \pm \gamma_0\gamma_4)\Phi \qquad \Phi = U_3\Psi \qquad \varkappa \leqslant m < \infty.$$

If one imposes the Galilean-invariant subsidiary condition $(p_0 + p_4)\Psi = m_0\Psi$ and puts $\varkappa = 0$, then equation (5.4) is reduced to the Levi-Leblond equation for the non-relativistic particle of spin $s = \frac{1}{2}$ (Levi-Leblond 1967). In this case (5.3) coincides with the operator which diagonalises the Levi–Leblond equation (Nikitin and Salogub 1975).

## 6. IR of the Poincaré group in the G(2) basis

The transition of the IR of the P(1,3) group to the basis of a two-dimensional Galilei group G(2) may be made by complete analogy with the reduction $P(1,4) \to G(3)$. Here we consider only the representations of the P(1,3) group, which correspond to time-like four-momenta. The generators of such a representation in a Shirokov–Foldy realisation (Shirokov 1954a, b, Foldy 1956) have the form (3.1) where $\mu, \nu = 0, 1, 2, 3$; $k, l = 1, 2, 3$. With the help of the transformation

$$P_\mu \to \tilde{P}_\mu = UP_\mu U^{-1} \qquad J_{\mu\nu} \to \tilde{J}_{\mu\nu} = UJ_{\mu\nu}U^{-1}$$

where

$$U = \exp\{(\mathrm{i}S_{3\alpha}p_\alpha/|p|)\tan^{-1}[|p|/(|P_0| + \epsilon p_3 + \varkappa)]\}, \qquad |p| = (p_1^2 + p_2^2)^{1/2} \qquad \alpha = 1, 2,$$

and the following replacement of the variables $\{p_1, p_2, p_3\} \to \{p_1, p_2, m\}$, where $m = \epsilon p_3 + (p_1^2 + p_2^2 + \varkappa^2)^{1/2}$, one obtains the generators of the Poincaré group in the G(2) basis:

$$\hat{P}_0 = \tfrac{1}{2}(\tilde{P}_0 + \tilde{P}_3) = \varkappa^2/2m + |p|^2/2m \qquad \hat{P}_\alpha = p_\alpha,$$
$$J_3 = \mathrm{i}[p_2(\partial/\partial p_1) - p_1(\partial/\partial p_2)] + S_{12} \qquad M = \epsilon m \tag{6.1}$$

$$G_\alpha^+ = \tilde{J}_{0\alpha} + \tilde{J}_{3\alpha} = -\mathrm{i}\epsilon m \frac{\partial}{\partial p_\alpha} \qquad |\varkappa| \leqslant m < \infty$$
$$G_\alpha^- = \tfrac{1}{2}(\tilde{J}_{0\alpha} - \tilde{J}_{3\alpha}) = \mathrm{i}[p_\alpha(\partial/\partial m) - \hat{P}_0(\partial/\partial p_\alpha)] - \epsilon(S_{\alpha\beta}p_\beta + S_{3\alpha}\varkappa)/m$$
$$K = \tilde{J}_{03} = -\mathrm{i}m(\partial/\partial m). \tag{6.2}$$

The operators (6.1) coincide with the 'kinematical group generators', which are used in the null-plane formalism (see e.g. Leutwyler and Stern 1968).

Using the results of § 3–§ 5, it is not difficult to make the transition into the G(2) basis of the representations of the P(1,3) algebra which corresponds to light-like and space-like four-momenta.

## 7. Connection between the Galilei and the Poincaré bases

We now consider the connection between the realisations of the generators of the P(1,4) group (corresponding to time-like five-momenta) in both the Galilei and Poincaré bases.

The generators of the P(1,4) group in the Poincaré basis (i.e. in the basis where the Casimir operators of the P(1,3) group are of diagonal type) have the form (Fushchich *et al* 1976, Nikitin *et al* 1976)

$$\begin{aligned}
&P_0 = E = (p^2+\bar{m}^2)^{1/2} \qquad P_a = p_a \qquad P_4 = \epsilon_4(\bar{m}^2+\varkappa^2)^{1/2}\\
&J_{ab} = \mathrm{i}[p_b(\partial/\partial p_a) - p_a(\partial/\partial p_b)] \qquad \epsilon_4 = \pm 1\\
&J_{0a} = -\mathrm{i}p_0(\partial/\partial p_a) - S_{ab}p_b/(E+\bar{m}) \qquad a, b = 1, 2, 3\\
&J_{04} = -\mathrm{i}E\{\epsilon_4(1-\varkappa^2/\bar{m}^2)^{1/2}, \partial/\partial\bar{m}\} - (\varkappa/\bar{m})(S_{4a}p_a/\bar{m})\\
&J_{4a} = \mathrm{i}p_a\{\epsilon_4(1-\varkappa^2/\bar{m}^2)^{1/2}, \partial/\partial\bar{m}\} - \mathrm{i}\epsilon\bar{m}(1-\varkappa^2/\bar{m}^2)^{1/2}\,\partial/\partial p_a\\
&\qquad + \frac{\varkappa p_a S_{4b}p_b}{m^2(E+m)} + \epsilon_4(1-\varkappa^2/\bar{m}^2)^{1/2}[S_{ab}p_b/(E+\bar{m})] + \frac{\varkappa S_{4a}}{\bar{m}}
\end{aligned} \tag{7.1}$$

where

$$\{A, B\} = AB + BA \qquad |\varkappa| \leqslant \bar{m} < \infty.$$

The generators (7.1) are Hermitian with respect to the scalar product

$$(\chi_1, \chi_2) = \sum_{s=|j-\tau|}^{j+\tau} \int_{\varkappa}^{\infty} \mathrm{d}\bar{m} \int \frac{\mathrm{d}^3p}{2E} \chi_1^{\dagger}(\boldsymbol{p}, \bar{m}, s, s_3)\chi_2(\boldsymbol{p}, \bar{m}, s, s_3).$$

As soon as the operators (7.1) and (3.17) realise the same IR $D^+(\varkappa, j, \tau)$ of the P(1,4) group, the equivalence transformation, which connects these two realisations, exists. In order to come from (7.1) to (3.17), we make the isometric transformation

$$P_\mu \to WP_\mu W^{-1} \qquad J_{\mu\nu} \to WJ_{\mu\nu}W^{-1} \tag{7.2}$$

and the following replacement of variables

$$p_a \to p_a \qquad \bar{m} \to \bar{m}(m, \boldsymbol{p}) \tag{7.3}$$

where

$$\begin{aligned}
&W = (1-\varkappa/\bar{m}^2)^{1/4}\exp[\mathrm{i}(S_{4a}p_a/p)(\theta_1-\theta_2)]\\
&\theta_1 = 2\tan^{-1}\{p/[E+\epsilon_4(\bar{m}^2-\varkappa^2)^{1/2}+\varkappa]\}\\
&\theta_2 = 2\tan^{-1}[\epsilon_4 p(\bar{m}^2-\varkappa^2)^{1/2}/(E+m)(m+\varkappa)]\\
&\bar{m} = (1/2m)[(m^2-\varkappa^2-p^2)^2+4m^2\varkappa^2]^{1/2}.
\end{aligned} \tag{7.4}$$

One can ensure by direct verification that the transformations (7.2)–(7.4) reduce the generators (7.1) into the Galilei basis (i.e. that the transformed generators coincide with (3.17) after substitution into (2.3)). We do not give the detailed calculations here because the transformations (7.2)–(7.4) may be represented as two consequent ones: namely, the transition from the Poincaré to the canonical basis (Nikitin *et al* 1976)

$$\begin{aligned} &P_\mu \to VP_\mu V^{-1} \qquad J_{\mu\nu} \to VJ_{\mu\nu}V^{-1} \\ &\bar{m} \to \bar{m}(p_4) = \epsilon_4(p_4^2 + \varkappa^2)^{1/2} \\ &V = (1-\varkappa^2/\bar{m}^2)^{1/4} \exp(\mathrm{i}S_{0a}p_a\theta_2/p) \end{aligned} \tag{7.5}$$

and then the transition from the canonical basis to the Galilei one (see § 3). So

$$W = U_1 V$$

where $V$ and $U_1$ are given by equations (7.5), (3.8), (3.14).

The transformation (7.2)–(7.4) may be used to establish the connection between the vectors in the Galilei and in the Poincaré bases. This connection is given by the equations:

$$\begin{aligned} &\phi(\boldsymbol{p}, m, s, s_3) = W\hat{P}_s\hat{P}_{s_3}P_{s'}P_{s'_3}\chi(\boldsymbol{p}, m(\bar{m}, \boldsymbol{p}), s, s_3) \\ &\chi(\boldsymbol{p}, m, s, s_3) = W^{-1}\tilde{P}_s\tilde{P}_{s_3}P_{s'}P_{s'_3}\phi(\boldsymbol{p}, \bar{m}(m, \boldsymbol{p}), s, s_3) \\ &m(\bar{m}, \boldsymbol{p}) = \epsilon_4(\bar{m}^2 - \varkappa^2)^{1/2} + (p^2 + \bar{m}^2)^{1/2} \\ &|j-\tau| \leqslant s,\ s' \leqslant j+\tau \qquad -s \leqslant s_3 \leqslant s \qquad -s' \leqslant s'_3 \leqslant s' \end{aligned}$$

where $P_s$, $P_{s_3}$, $\hat{P}_s$, $\hat{P}_{s_3}$, $\tilde{P}_s$, $\tilde{P}_{s_3}$, are the projectors into the subspace with the corresponding fixed value of $s$ and $s_3$.

$$P_s = \prod_{\tilde{s}\neq s} \frac{S^2 - \tilde{s}(\tilde{s}+1)}{s(s+1) - \tilde{s}(\tilde{s}+1)} \qquad P_{s_3} = \prod_{\tilde{s}_3 \neq s_3} \frac{S_3 - \tilde{s}_3}{s_3 - \tilde{s}_3} \tag{7.6}$$

$$\hat{P}_s = W^{-1}P_sW \qquad \hat{P}_{s_3} = W^{-1}P_{s_3}W \qquad \tilde{P}_{s_3} = WP_{s_3}W^{-1} \qquad \tilde{P}_s = WP_sW^{-1}.$$

$\hat{P}_s$, $\hat{P}_{s_3}$, $\tilde{P}_s$, $\tilde{P}_{s_3}$ may be obtained from (7.6) by the substitution

$$\begin{aligned} S_a \to \hat{S}_a = W^{-1}S_aW &= S_a \cos\tilde{\theta} + (p_aS_bp_b/p^2)(1-\cos\tilde{\theta}) \\ &\quad + \frac{1}{p}\epsilon_{abc}p_bS_{4c}\sin\tilde{\theta} \qquad \tilde{\theta} = \theta_1 - \theta_2 \end{aligned}$$

$$S_a \to \tilde{S}_a = WS_aW^{-1} = S_a\cos\tilde{\theta} + (p_aS_bp_b/p^2)(1-\cos\tilde{\theta}) - (1/p)\epsilon_{abc}p_bS_{4c}\sin\tilde{\theta}.$$

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## References

Aghassi J J, Roman P and Santilli R M 1970 *J. Math. Phys.* **11** 2297–305
Bargman V 1954 *Ann. Math.* **59** 1–46
Barrabes K and Henry J 1976 *J. Phys. A: Math. Gen.* **9** 1428–34
Castell L 1967 *Nuovo Cim.* **49** 285–94

Elizalde E and Gomish J 1978 *J. Math. Phys.* **19** 1790–7
Fedorchuck V M 1978 *Mathematical Institute, Kiev, preprint* 78–18
Foldy L L 1956 *Phys. Rev.* **102** 568–74
Fushchich V I 1970 *Teor. Mat. Fiz.* **4** 360–7
Fushchich V I and Krivsky I Yu 1968a *Nucl. Phys.* B **7** 79–87
—— 1968b *Institute for Theoretical Physics, Kiev, preprint* 68–72
—— 1969 *Nucl. Phys.* B **14** 537–44
Fushchich V I, Nikitin A G and Yurik I I 1976 *Mathematical Institute, Kiev, preprint* 75–85
Leutwyler H and Stern J 1968 *Ann. Phys.* **112** 94–164
Levi-Leblond J-M 1967 *Comm. Math. Phys.* **6** 286–311
Nikitin A G, Fushchich V I and Yurik I I 1976 *Teor. Mat. Fiz.* **26** 202–20 (transl. *Teoret.* and *Math. Phys.* **26** 138–47)
Nikitin A G and Salogub V A 1975 *Ukr. Fiz. J.* **20** 1730–2
Shirokov Yu M 1954a *Dokl. Akad. Nauk USSR* **94** 857–61
—— 1954b *Dokl. Akad. Nauk USSR* **99** 737–9